STUDIES OF THE UNBIASED FIR FILTER FOR THE TIME ERROR MODEL IN APPLICATIONS TO GPS-BASED TIMEKEEPING[1]

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Abstract

The unbiased FIR filter is investigated to estimate the time interval error (TIE) K-D polynomial model of a local clock in GPS-based timekeeping in the presence of uniformly distributed sawtooth noise. An estimation algorithm is proposed and applied to the GPS-based TIE measurements of a crystal clock without using the sawtooth correction. Based upon this, we show that the TIE estimates fit actual values with an uncertainty of the GPS time. It is also demonstrated that estimates of the fractional frequency offset fit the measurements with the frequency shifts present in the reference rubidium source and the 1 PPS signal of a GPS receiver used.

INTRODUCTION

Fast and accurate estimation and steering of a local clock performance is still a key problem in GPS-based timekeeping. Here, the time interval error (TIE) between the GPS time and the local clock time is measured in the presence of noise induced by the GPS timing receiver. The TIE is then estimated and the feedback system causes the local clock to track the GPS time. The standard deviation of the noise using commercially available receivers is about 30 ns, can reach 10-20 ns [**1**], and may be improved by removal of systematic errors to no less than 3-5 ns [**1,2**]. Having such a large noise, the measured data cannot be used straightforwardly for locking, and a TIE tracking filter is applied. To obtain filtering in an optimum way, a TIE model of a local clock must be known for the filter memory. Such a model was proposed in [**4**] as a third degree (3-D) Taylor polynomial and is now basic in timekeeping, being practically proved. In the discrete time, the model may be written as

$$x(n) = x_0 + y_0\Delta n + \frac{D}{2}\Delta^2 n^2 + w(n,\tau), \tag{1}$$

where $n = 0, 1, ...$; $\Delta = t_n - t_{n-1}$ is a sample time; t_n is a discrete time; x_0 is an initial time error; y_0 is an initial fractional frequency offset of a local clock from the reference frequency; D is an initial linear fractional frequency drift rate; and $w(n,\tau)$ is a random component caused by the oscillator colored Gaussian noise and environmental influences. In GPS-based measurements, (1) is observed via the mixture

$$\xi(n) = x(n) + v(n), \tag{2}$$

in which $v(n)$ is a noisy component induced at receiver (the noise of a measurement set is usually negligible). It is also typical for GPS-based measurements that the noise mean square values relate as $\langle w^2(n,\tau)\rangle <<$

[1]These studies were supported in part by the CONACyT Project, J28818-A

$\langle v^2(n)\rangle$. Therefore, $w(n,\tau)$ may be neglected in the averaging procedure (it cannot be discarded in the Kalman filter).

In modern receivers, such as the Motorola M12+ (see **[3]**), and SynPaQ III GPS Sensor, a random variable $v(n)$ is uniformly distributed owing to a principle of the 1 PPS (one pulse per second) signal formation. Figure 1a shows a typical sawtooth structure of $v(n)$ in a short time; that is the modulo $2\sigma_{max}$ Brownian TIE associated with a phase of a receiver local oscillator, where σ_{max} is the error bound (in SynPaQ III, it is about 50 ns). In long-term measurements, $v(n)$ exhibits nonstationary excursions (Figure 1b) caused by uncertainties of a GPS time at receiver (see **[5]**) with different satellite sets in a view. If a single-channel receiver is used, the excursions have a day's periodicity **[5]**. With multichannel receivers, they are typically nonperiodic (Figure 1b). In the latter case, a long-term noise histogram evolves, by increasing the data, from uniform to normal and v_n may be approximated by a mean-zero, $\langle v(n)\rangle = 0$, stationary Gaussian noise. If the sawtooth correction is used, then the noise $v(n)$ becomes near Gaussian in a short time.

To estimate optimally x_0, y_0, and D for different clocks (atomic and crystals), assuming the Gaussian nature of $v(n)$ with a known variance σ_v^2, several filtering algorithms have been examined over a couple of decades. For the state space equivalent of (1), the problem was formulated by Allan and Barnes in **[6]** to apply Kalman filtering. Several solutions were then given in **[7-9]**. Thereafter, various Kalman algorithms were examined by the authors in **[10-14]** for different estimation purposes. These and other applications of Kalman filters to time scales were recently outlined in **[15]**. The problem with the Kalman filter arises when noise is non-Gaussian, as in Figure 1, for example. To overcome this, advanced Kalman algorithms were proposed in **[16,17]**, being, however, not yet adopted for time scales. Alternatively, finite impulse response (FIR) filters are used that allow for noise of an arbitrary distribution. In contrast to the infinite impulse response (IIR) structures (including Kalman filters), FIR structures have inherent properties, such as a bounded input/bounded output (BIBO) stability and robustness against temporary model uncertainties and round-off errors **[20]**. They may be used independently or combined with Kalman filters **[12,19]**. A general optimal FIR filtering algorithm with embedded unbiasedness for state space models was recently proposed in **[21]**. Especially for GPS-based timekeeping and a linear TIE model, an unbiased FIR filter was designed and studied in **[22]**. In this paper, we investigate a new unbiased FIR filter and filtering algorithm intended for real-time estimation of the TIE K-D polynomial model in the presence of noise of an arbitrary symmetric distribution.

THE TIE POLYNOMIAL MODEL

In view of $\langle w_n^2\rangle << \langle v_n^2\rangle$, the noise-free and time-invariant TIE model projects ahead on a horizon of N points from the start point $n=0$ with the K-D Taylor polynomial

$$x_1(n) = \sum_{p=0}^{K} \mathrm{x}_{p+1}\frac{\Delta^p n^p}{p!} = x_1(0) + x_2(0)\Delta n + \frac{x_3(n)}{2}\Delta^2 n^2 + \frac{x_4(n)}{6}\Delta^3 n^3 ..., \tag{3}$$

By extending the time derivatives of $x_1(n)$ to the Taylor series, the clock model and its observation (2) become, respectively,

$$\lambda(n) = \mathbf{B}(n)\lambda(0), \tag{4}$$

$$\xi(n) = \mathbf{C}\lambda(n) + \mathbf{v}(n), \tag{5}$$

where $\lambda(n) = [x_1(n)x_2(n)...x_{K+1}(n)]^T$ is a vector $[(K+1)\times 1]$ of clock functions with the components approximately calculated for $l > 1$, $l \in [0, K]$, by

$$x_l(n) = \frac{1}{\Delta}[x_{l-1}(n) - x_{l-1}(n-1)], \tag{6}$$

and a transition matrix $[(K+1)\times(K+1)]$ is

$$\mathbf{B}(n)=\begin{bmatrix} 1 & \Delta n & \Delta^2 n^2/2 & ... & (\Delta n)^K/K! \\ 0 & 1 & \Delta n & ... & (\Delta n)^{K-1}/(K-1)! \\ 0 & 0 & 1 & ... & (\Delta n)^{K-2}/(K-2)! \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}. \tag{7}$$

The most common situation that may be assumed in timekeeping is when all or several clock states are observable, by (5), via M (independent or dependent) measurements in presence of correlated or uncorrelated noises. Then the observation vector is $\xi(n)=[\xi_1(n)\xi_2(n)...\xi_M(n)]^T$ and

$$\mathbf{C}=\begin{bmatrix} c_{11} & 0 & ... & 0 & ... & 0 \\ 0 & c_{22} & ... & 0 & ... & 0 \\ ... & ... & ... & ... & ... & 0 \\ 0 & 0 & ... & c_{MM} & ... & 0 \end{bmatrix} \tag{8}$$

is the measurement matrix $[M\times(K+1)]$ with, typically, unit components. Finally, in view of Figure 1, the noise vector $\mathbf{v}(n)=[v_1(n)v_2(n)...v_M(n)]^T$ contains correlated or uncorrelated components that are not obligatory Gaussian. In the following, we will assume that the TIE model (4) has $K+1$ states and is observable via (5) in presence of sawtooth noise (Figure 1a) of unknown structure (Figure 1b) caused by GPS time uncertainty. We will also assume that $M=K+1$, $\mathbf{C}$ is an identity matrix, and the noise $\mathbf{v}(n)$ is mean zero, $\langle\mathbf{v}(n)\rangle=0$, and symmetrically distributed with known covariance $\langle\mathbf{v}^T(n)\mathbf{v}(n)\rangle$.

AN UNBIASED FIR FILTER

Utilizing N points of the nearest past, the FIR estimate $\hat{\lambda}(n)=[\hat{x}_1(n)\hat{x}_2(n)...\hat{x}_{K+1}(n)]^T$ at n-th point is given by, for $M=K+1$,

$$\hat{\lambda}(n)=\sum_{i=0}^{N-1}\mathbf{H}(i)\xi(n-i) \tag{9}$$

$$=\mathbf{q}(n)\mathbf{\Gamma}, \tag{10}$$

$$=\mathbf{d}(n)\mathbf{\Gamma}+\mathbf{r}(n)\mathbf{\Gamma}, \tag{11}$$

where the matrix $[(K+1)\times(K+1]$ of unknown FIRs is

$$\mathbf{H}(i)=\begin{bmatrix} h_K(i) & 0 & ... & 0 \\ 0 & h_{K-1}(i) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & h_0(i) \end{bmatrix}, \tag{12}$$

in which the l-th FIR has inherent properties: $h_l(i)=\begin{cases} h_l(i), 0\le i\le N-1 \\ 0, \quad \text{otherwise}\end{cases}$ and $\sum_{i=0}^{N-1}h_l(i)=1$. The estimates vector and its deterministic and random constituents, all of $[(K+1)\times N]$, are, respectively,

$$\mathbf{q}(n)=[\mathbf{H}(0)\xi(n) \quad \mathbf{H}(1)\xi(n-1) \quad ... \quad \mathbf{H}(N-1)\xi(n-N+1)] \tag{13}$$

$$\mathbf{d}(n)=[\mathbf{H}(0)\mathbf{C}\lambda(n) \quad \mathbf{H}(1)\mathbf{C}\lambda(n-1) \quad ... \quad \mathbf{H}(N-1)\mathbf{C}\lambda(n-N+1)], \tag{14}$$

$$\mathbf{r}(n)=[\mathbf{H}(0)\mathbf{v}(n) \quad \mathbf{H}(1)\mathbf{v}(n-1) \quad ... \quad \mathbf{H}(N-1)\mathbf{v}(n-N+1)], \tag{15}$$

and an auxiliary unit matrix ($N \times 1$) is $\mathbf{\Gamma} = [1 \quad 1 \quad ...1]^T$. The mean square error (MSE) of $\hat{\lambda}(n)$ is

$$J(n) = \langle[\lambda(n) - \hat{\lambda}(n)]^T[\lambda(n) - \hat{\lambda}(n)]\rangle = [\lambda(n) - \mathbf{d}(n)\mathbf{\Gamma}]^T[\lambda(n) - \mathbf{d}(n)\mathbf{\Gamma}] + \langle[\mathbf{r}(n)\mathbf{\Gamma}]^T[\mathbf{r}(n)\mathbf{\Gamma}]\rangle, \tag{16}$$

in which the estimate bias and variance are, respectively,

$$\text{bias}[\hat{\lambda}(n)] = \lambda(n) - \mathbf{d}(n)\mathbf{\Gamma}, \tag{17}$$

$$\text{var}[\hat{\lambda}(n)] = \langle[\mathbf{r}(n)\mathbf{\Gamma}]^T[\mathbf{r}(n)\mathbf{\Gamma}]\rangle. \tag{18}$$

Generic Coefficients for the FIR of an Unbiased Filter

The necessary and sufficient condition for the unbiased estimate follows straightforwardly from (17); that is,

$$\lambda(n) = \mathbf{d}(n)\mathbf{\Gamma}, \tag{19}$$

providing the rule to derive FIRs for the clock states, namely:

$$\begin{bmatrix} x_1(n) \\ x_2(n) \\ ... \\ x_{K+1}(n) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_K^T\lambda_1(n) \\ \mathbf{W}_{K-1}^T\lambda_2(n) \\ ... \\ \mathbf{W}_0^T\lambda_{K+1}(n) \end{bmatrix}, \tag{20}$$

where

$$\mathbf{W}_l = [h_l(0)h_l(1)...h_l(N-1)]^T, \tag{21}$$

$$\lambda_{K+1-l}(n) = \begin{bmatrix} x_{K+1-l}(n) \\ x_{K+1-l}(n-1) \\ ... \\ x_{K+1-l}(n-N+1) \end{bmatrix}. \tag{22}$$

For the clock $(K+1-l)$-th state, (20) thus yields a relation

$$x_{K+1-l}(n) = \mathbf{W}_l^T\lambda_{K+1-l}(n). \tag{23}$$

By invoking (6) and then expressing the components in (20) with the l-D polynomial, by (3), we arrive at the unbiasedness (or deadbeat) constraint

$$\mathbf{F}\mathbf{W}_l = \mathbf{G}, \tag{24}$$

in which $\mathbf{G} = [1 \quad 0 \quad ... \quad 0]^T$ is of $[(l+1) \times 1]$ and

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & 1 & 2 & ... & N-1 \\ 0 & 1 & 2^2 & ... & (N-1)^2 \\ ... & ... & ... & ... & ... \\ 0 & 1 & 2^l & ... & (N-1)^l \end{bmatrix}. \tag{25}$$

We notice that the constraint (24) is inherent for any other unbiased estimators, e.g., for the minimum variance unbiased (MVU) and best linear unbiased estimator (BLUE). Now, the components in (21) may also be substituted by the l-D polynomial (that is what Kalman claims for optimal filtering)

$$h_l(i) = \sum_{j=0}^{l} a_{jl}i^j, \tag{26}$$

where a_{jl} are still unknown coefficients. Embedded (26), the constraint (24) becomes

$$\mathbf{DA} = \mathbf{G}, \tag{27}$$

where the FIR coefficients matrix is $\mathbf{A} = [a_{0l} a_{1l} ... a_{ll}]^T$, and an auxiliary quadratic matrix $[(l+1) \times (l+1)]$ is given by

$$\mathbf{D} = \begin{bmatrix} d_0 & d_1 & d_2 & ... & d_l \\ d_1 & d_2 & d_3 & ... & d_{l+1} \\ d_2 & d_3 & d_4 & ... & d_{l+2} \\ ... & ... & ... & ... & ... \\ d_l & d_{l+1} & d_{l+2} & ... & d_{2l} \end{bmatrix}, \tag{28}$$

where the generic component $d_m = \sum_{i=0}^{N-1} i^m$, $m \in [0, 2l]$, is determined by the Bernoulli polynomials (Appendix A). An analytic solution of (27) yields the generic coefficients for (26)

$$a_{jl} = (-1)^j \frac{M_{(j+1)1}}{|\mathbf{D}|}, \tag{29}$$

in which $|\mathbf{D}|$ and $M_{(j+1)1}$ are the determinant and minor of (26), respectively. Determined a_{jl} and $h_l(n)$, the unbiased FIR estimate of the clock $(K+1-l)$-th state is given to be

$$\hat{x}_{K+1-l}(n) = \sum_{i=0}^{N-1} h_l(i) \xi_{K+1-l}(n-i) \tag{30}$$

$$= \mathbf{W}_l^T \mathbf{\Xi}_{K+1-l}(n), \tag{31}$$

where

$$\mathbf{\Xi}_{K+1-l}(n) = \begin{bmatrix} \xi_{K+1-l}(n) \\ \xi_{K+1-l}(n-1) \\ ... \\ \xi_{K+1-l}(n-N+1) \end{bmatrix}. \tag{32}$$

Estimate Noise

The estimate noise variance (18) may now be rewritten as

$$\text{var}[\hat{\lambda}(n)] = \langle [\mathbf{r}(n)\mathbf{\Gamma}]^T [\mathbf{r}(n)\mathbf{\Gamma}] \rangle = \sum_{k=0}^{K} \mathbf{W}_{K-k}^T \mathbf{R}_{k+1}(n) \mathbf{W}_{K-k}, \tag{33}$$

$$\mathbf{R}_l(n) = \begin{bmatrix} R_l(n,n) & R_l(n,n-1) & ... & R_l(n,n-N+1) \\ R_l(n-1,n) & R_l(n-1,n-1) & ... & R_l(n-1,n-N+1) \\ ... & ... & ... & ... \\ R_l(n-N+1,n) & R_l(n-N+1,n-1) & ... & R_l(n-N+1,n-N+1) \end{bmatrix}. \tag{34}$$

where the autocorrelation matrix $\mathbf{R}_l(n)$ is specified by (33) with a generic component $R_l(i,j) = \langle v_l(i) v_l(j) \rangle$, $i, j \in [n, n-N+1]$. Accordingly, the estimate variance associated with the estimate (31) calculates

$$\sigma_{K+1-l}^2(n) = \mathbf{W}_l^T \mathbf{R}_{K+1-l}(n) \mathbf{W}_l. \tag{35}$$

It is important that, by large N, the sawtooth noise becomes delta-correlated. This degenerates (34) to the diagonal form with the components $R_l(i,i) = \sigma_{vl}^2$ and (35) is written as

$$\sigma_{K+1-l}^2 = \sigma_{v(K+1-l)}^2 \mathbf{W}_l^T \mathbf{W}_l, \tag{36}$$

where $\sigma_{v(K+1-l)}^2$ is a variance of the noise perturbing the $(K+1-l)$-th clock state.

Let us note that, in different algorithms, the components of the noise vector $\mathbf{v}(n)$ may be caused by different or equal physical sources. Thus, they may demonstrate different powers of correlation and dependence.

APPLICATIONS TO THE CLOCK TIE POLYNOMIAL MODEL

In applications, K in (3) is identified for the filter memory on a horizon $[0, N-1]$) by the clock precision. Typically, it is assumed that (3) fits cesium and hydrogen maser clocks with $K \in [0,1]$, and crystal and rubidium clocks with $K \in [1,2]$. However, $K = 3$ may be required for low-precision crystal clocks. Below we give the unbiased FIRs for all these cases.

Low-Order FIRs for the Optimally Unbiased Filters

Setting $l = 0,1,2,3$ and using the coefficient d_m (Appendix), we first calculate the FIR coefficients (29). Then the relevant FIRs (26) may readily be written as

$$h_0(i) = \frac{1}{N}, \tag{37}$$

$$h_1(i) = \frac{2(2N-1) - 6i}{N(N+1)}, \tag{38}$$

$$h_2(i) = \frac{3(3N^2 - 3N + 2) - 18(2N-1)i + 30i^2}{N(N+1)(N+2)}, \tag{39}$$

$$h_3(i) = \frac{8(2N^3 - 3N^2 + 7N - 3) - 20(6N^2 - 6N + 5)i}{N(N+1)(N+2)(N+3)} + \frac{120(2N-1)i^2 - 140i^3}{N(N+1)(N+2)(N+3)}. \tag{40}$$

The constant FIR (37) corresponds to simple averaging and is optimal in a sense of minimum produced noise. This FIR is practically proved to be reasonable in GPS-based common view measurements **[5]**. The linear FIR (38) was also derived in **[22]** by using linear regression to compensate a bias of simple averaging. This filter demonstrates an intermediate error between the 2-D and 3-D Kalman filters in application to the crystal and rubidium clocks **[24]**. Its kernel starts with a maximum $h_2(0) = \left.\frac{2(2N-1)}{N(N+1)}\right|_{N>>1} \cong \frac{4}{N} > 0$ and goes to a minimum $h_2(N-1) = -\left.\frac{2(N-2)}{N(N+1)}\right|_{N>>1} \cong -\frac{2}{N} < 0$, having zero at $n_0 = \frac{2N-1}{3}$. Its special peculiarity is $r = \lim\limits_{N\to\infty} \frac{h_2(0)}{h_2(N-1)} = -2$ that allows one to synthesize a FIR, by saving $r = -2$ for arbitrary N. It may be shown that the FIR synthesized in such a way is equal to that derived in **[23]** for the 1-step linear prediction on a horizon $[1, N]$.

Noise Power Gains

The noise power gain corresponding to the l-D FIR is specified, by (36), to be $g_l(N) = \sigma^2_{K+1-l}/\sigma^2_{v(K+1-l)} = \mathbf{W}_l^T\mathbf{W}_l$. Its values associated with (37)-(40) are given below, respectively,

$$g_0(N) = \frac{1}{N}, \tag{41}$$

$$g_1(N) = \frac{2(2N-1)}{N(N+1)}, \tag{42}$$

$$g_2(N) = \frac{3(3N^2-3N+2)(N^2+3N+2)}{N(N+1)^2(N+2)^2}, \tag{43}$$

$$g_3(N) = \frac{8(2N^3-3N^2+7N-3)}{N(N+1)(N+2)(N+3)}. \tag{44}$$

Relations (41)-(44) manifest that unbiasedness is achieved at an increase of noise. Indeed, the curves for $l > 0$ trace above the lower bound $1/\sqrt{N}$ associated with simple averaging ($l = 0$) that produces minimum noise (among all filters). It also follows that, by large $N > 100$, the noise gain is performed approximately by $\sqrt{g_l(N)} \cong (l+1)/\sqrt{N}$ and, thus, traces below the upper bound; that is,

$$\sqrt{g_l(N)} \le \begin{cases} (l+1)/\sqrt{N}, N \ge (l+1)^2 \\ 1, N < (l+1)^2. \end{cases} \tag{45}$$

ESTIMATING THE TIE MODEL WITH A SINGLE GPS TIMING RECEIVER

We now consider the most widely used practical case when the measurement $\xi_1(n)$ of a TIE $x_1(n)$ is obtained with a single multichannel GPS timing receiver and the sawtooth correction is not used. An estimation algorithm is shown in Figure 2 for the K-D polynomial TIE model (3). The first clock state estimate $\hat{x}_1(n)$ is provided with $h_K(n)$ by (30) at a horizon $[0, N_K - 1]$. The observation $\xi_2(n)$ for the second state $x_2(n)$ is then formed, using (6), as a discrete time derivative of $\hat{x}_1(n)$. Accordingly, $\hat{x}_2(n)$ is achieved with $h_{K-1}(n)$ at a horizon $[0, N_{K-1} - 1]$. Inherently, the first accurate value of $\hat{x}_2(n)$ appears at $(N_K + N_{K-1} - 2)$th point. Finally, the last state estimate $\hat{x}_{K+1}(n)$ is calculated with $h_0(n)$ at a horizon $[0, N_0 - 1]$ using $\xi_{K+1}(n)$ that is formed in the same manner as $\xi_2(n)$. The first correct value of $\hat{x}_{K+1}(n)$ appears at $(N_K + N_{K-1} + ... + N_0 - K - 1)$th point.

Below, as an example of application, we use this algorithm to estimate the TIE model of a crystal clock embedded to the Stanford Frequency Counter SR620. The measurement is done with SynPaQ III and SR620 for $\Delta = 1$ s (GPS-measurement). Simultaneously, to get a reference noiseless trend, the TIE of the same crystal clock is measured, by SR625, for the rubidium clock (Rb-measurement). Initial time and frequency shifts between two measurements are then eliminated statistically and transition to $\Delta = 10$ s is provided by the data thinning in time. A horizon length N_l for each FIR filter is set to obtain the estimates with minimum MSEs.

Several-Hour Measurement and Estimation

In this experiment, a short-term measurement of the TIE has been done during several hours. The TIE model was then identified in the sense of a minimum MSE to be quadratic. Accordingly, we set $K = 2$, estimate three clock states, and compare the results for the Rb-measurement. Figure 3 illustrates the estimates $\hat{x}_1(n)$ and $\hat{x}_2(n)$. We notice that, inherently, a several-hour measurement does not allow for a proper estimate of the third clock state $\hat{x}_3(n)$. It is neatly seen (Figure 3*a*) that the estimate of TIE $\hat{x}_1(n)$ follows the mean value of the GPS-measurement and that its offset from the Rb-measurement is caused mostly by the GPS time uncertainty. A maximum estimate error of about 60 ns has appeared in the time span between the 8th and 9th hours when a time shift in the IPPS signal was fixed. Figure 3*b* shows that the estimate $\hat{x}_2(n)$ of the fractional frequency offset fits well the first time derivative of the Rb-measurement. However, in contrast to (Figure 3*a*), here the frequency shift of about 3×10^{-11} has occurred in the span between the 7th and 8th hours and no appreciable error has been fixed in the range of large time shift (between the 8th and 9th hours).

Long-Term Measurement and Estimation

In this experiment, we watched for the same crystal clock over about 2.5 days. The measurements are inherently fixed oscillations associated with a day's variation in temperature (Figure 4*a*). We notice that, like the previous case, $\hat{x}_1(n)$ fits the reference Rb-measurement with the error caused by the GPS time uncertainty. The estimate $\hat{x}_2(n)$ tracks the mean value of the first time derivative of $\hat{x}_1(n)$ (Figure 4*b*). Finally, $\hat{x}_2(n)$ fits well the first time derivative of the Rb-measurement (Figure 4*c*). And, again, a 2.5-day measurement does not allow for estimating $\hat{x}_3(n)$ with a sufficient accuracy.

CONCLUSION

In this paper, we studied an optimally unbiased FIR filter of the TIE polynomial model of a local clock. In contrast to the standard Kalman filter, the proposed solution does not require the TIE measured process to be Gaussian and does not involve any knowledge about noise in the algorithm. Therefore, the algorithm seems to be simpler for engineering applications. The filter is asymptotically optimal, since a variance of the produced noise reduces as a reciprocal of the horizon length N. Let us add that timekeeping operates typically with large horizons, $N >> 1$. An application of the proposed algorithm to the GPS-based measurement of a crystal clock showed that the estimate of TIE $\hat{x}_1$ fits actual values with an uncertainty of the GPS time. Therefore, such a filter may also be employed as an estimator of the GPS time uncertainty in hybrid structures. The estimate of the fractional frequency offset $\hat{x}_2$ fits the reference Rb-measurement with high accuracy that is limited by frequency shifts in the 1 PPS signal of the GPS receiver and in the reference signal. Thus, such timing receivers may efficiently be used in remote measurements of frequency offsets of local oscillators instead of expensive quantum sources.

APPENDIX: THE COEFFICIENTS OF MATRIX (25)

The coefficients for (28) are calculated by

$$d_m = \sum_{i=0}^{N-1} i^m = \frac{1}{m+1} \left[B_{m+1}(N) - B_{m+1} \right],$$

where $B_n(x)$ is the Bernoulli polynomial and $B_n = B_n(0)$ is the Bernoulli coefficient. For low orders, $B_n(x)$ may be found in reference books. For high orders, the following recurrent relation is valid:

$$B_n(x) = n \int B_{n-1}(x)dx + B_n.$$

Several low order coefficients d_m are given below

$$d_0 = N,\ \ d_1 = \frac{N(N-1)}{2},\ \ d_2 = \frac{N(N-1)(2N-1)}{6},\ \ d_3 = \frac{N^2(N-1)^2}{4},$$

$$d_4 = \frac{N(N-1)(2N-1)(3N^2-3N-1)}{30},\ \ d_5 = \frac{N^2(N-1)^2(2N^2-2N-1)}{12},$$

$$d_6 = \frac{N(N-1)(2N-1)(3N^4-6N^3+3N+1)}{42}.$$

For a large horizon, $N >> 1$, the coefficients d_m may be calculated by $d_m|_{N>>1} \cong \frac{N^{m+1}}{m+1}$.

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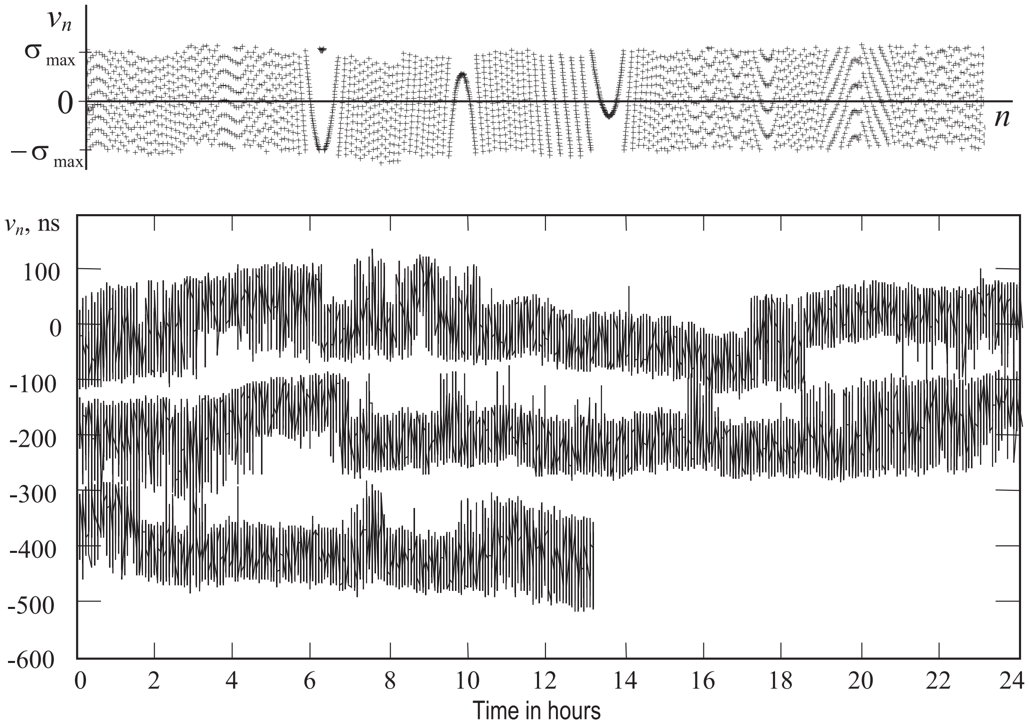

Figure 1: Typical TIE sawtooth noise structure induced by a GPS timing sensor SynPaQ III: (a) short-term, 1 s measurements and (b) long-term, 10 s measurements with a day's data shifted by 200 ns.

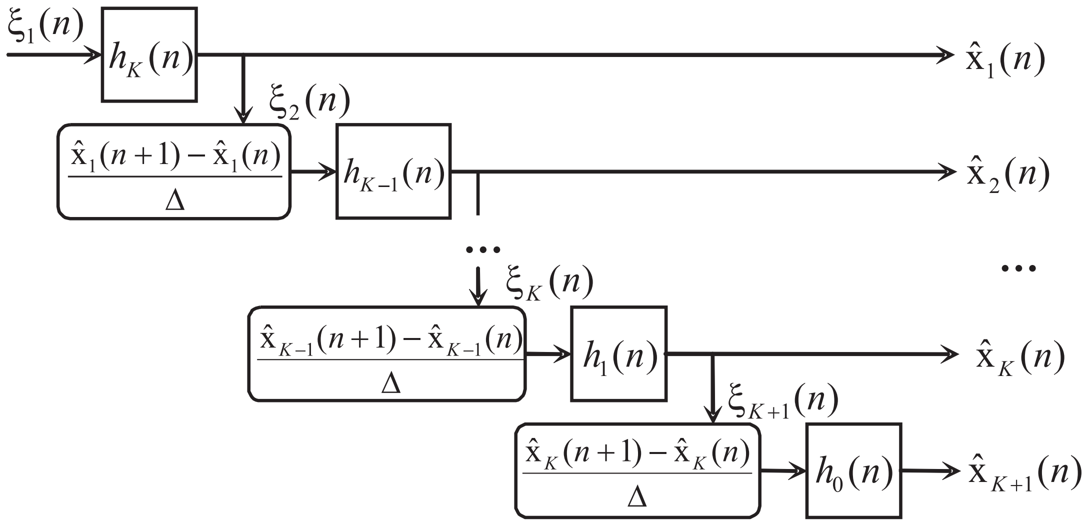

Figure 2: Structure of the unbiased FIR filtering algorithm for the K-D TIE polynomial model observable with a single GPS timing receiver.

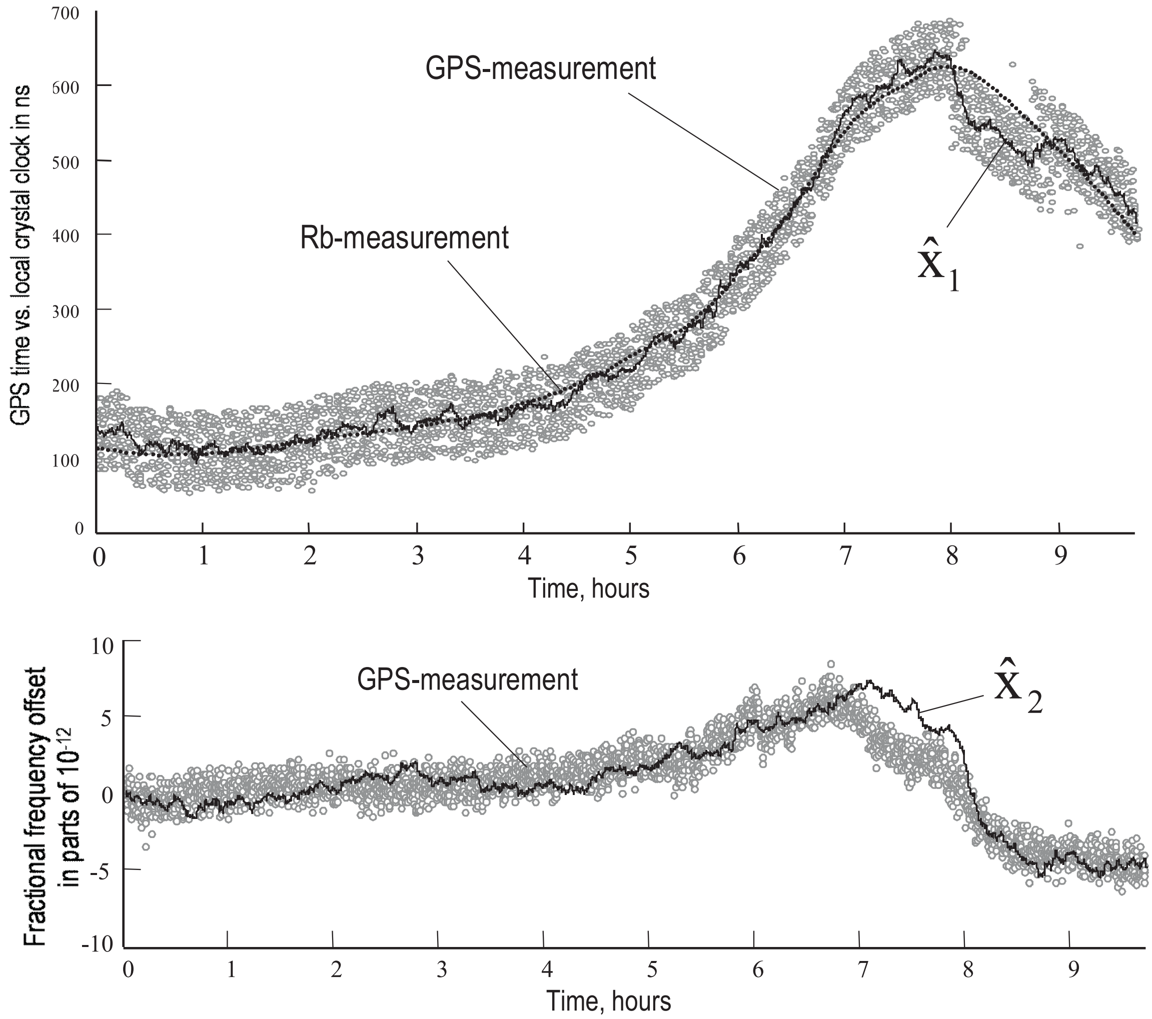

Figure 3: Short-time measurement and estimation of the crystal clock TIE model without the sawtooth correction: (*a*) GPS-measurement of the TIE (points), Rb-measurement of the TIE (dotted), and quadratic unbiased FIR estimate of TIE $\hat{x}_1(n)$ (solid); (*b*) Rb-measurement of the fractional frequency offset (points) and its GPS-based linear unbiased FIR estimate $\hat{x}_2(n)$ (solid) (Note: the scale here in 10^{-11})

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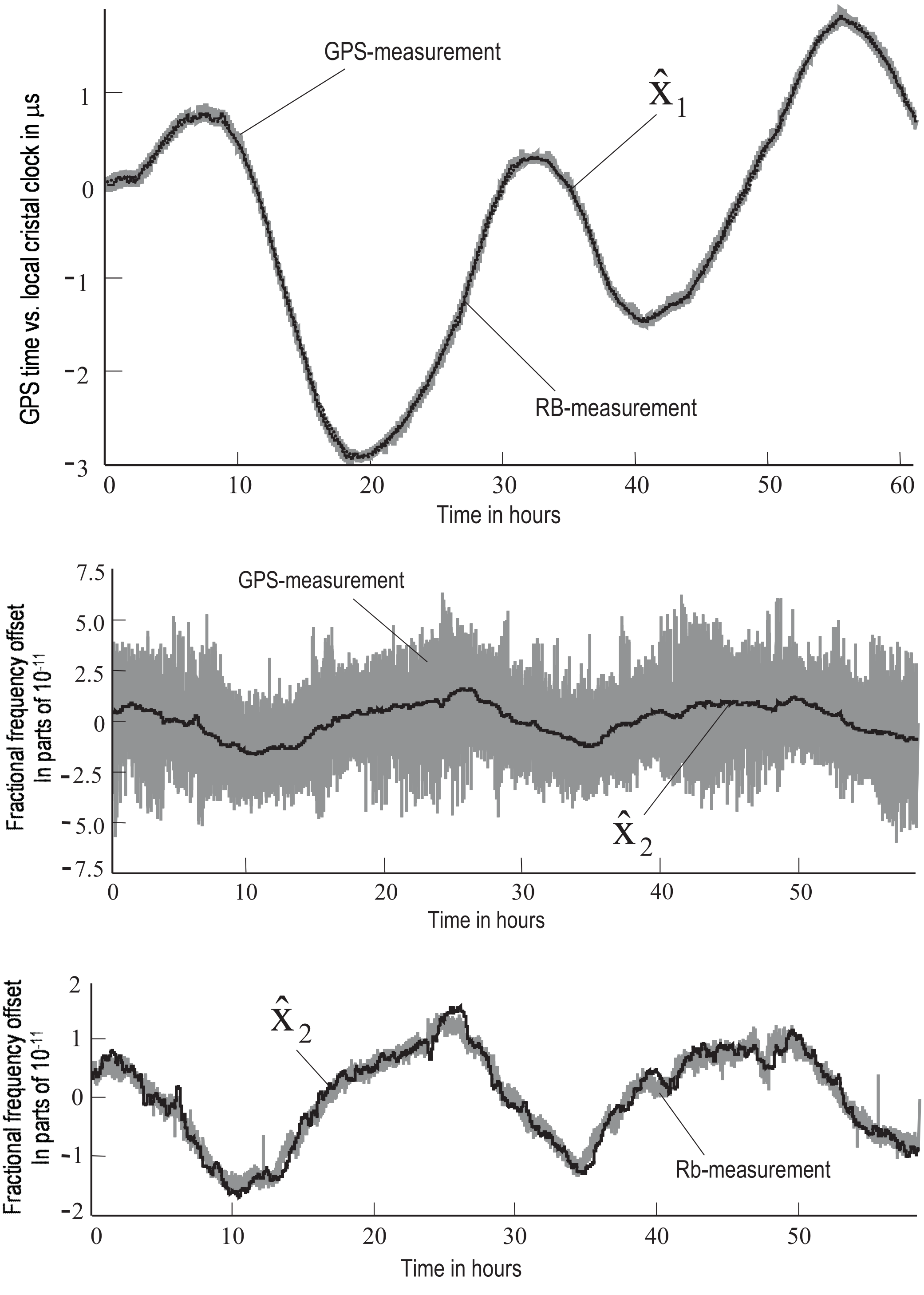

Figure 4: Long-term measurement and estimation of the crystal clock TIE without the sawtooth correction: (a) GPS-measurement (points), Rb-measurement (dotted), and quadratic unbiased FIR estimate of TIE $\hat{x}_1(n)$ (solid); (b) observation $\xi_2(n)$ (light) and linear unbiased FIR estimate of the fractional frequency offset $\hat{x}_2(n)$ (bold) (Note: the scale here in 10^{-10}) ; (c) Rb-measurement (light) and estimate $\hat{x}_2(n)$ (solid) (Note: the scale here in 10^{-10}).